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Generic replica symmetric field-theory for short range Ising spin glasses

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Abstract. Symmetry considerations and a direct, Hubbard-Stratonovich type, derivation are used to construct a replica field-theory relevant to the study of the spin glass transition of short range models in a magnetic field. A mean-field treatment reveals that two different types of transitions exist, whenever the replica number n is kept larger than zero. The Sherrington-Kirkpatrick critical point in zero magnetic field between the paramagnet and replica magnet (a replica symmetric phase with a nonzero spin glass order parameter) separates from the de Almeida-Thouless line, along which replica symmetry breaking occurs. We argue that for studying the de Almeida-Thouless transition around the upper critical dimension $d = 6$, it is necessary to use the generic cubic model with all the three bare masses and eight cubic couplings. The critical role n may play is also emphasized. To make perturbative calculations feasible, a new representation of the cubic interaction is introduced. To illustrate the method, we compute the masses in one-loop order. Some technical details and a list of vertex rules are presented to help future renormalisation-group calculations.

PACS. 75.10.Nr Spin-glass and other random models – 05.70.Jk Critical point phenomena

1 Introduction

The Ising spin glass is the simplest model still incorporating all the complexity that more sophisticated disordered systems show up. As such, it has become widely studied in the last decades. We focus our attention here to the case where Ising spins interact *via* Gaussian-distributed pair interactions [1]. A huge amount of literature has been accumulated since the seminal paper of Edwards and Anderson [2], nevertheless the most important problems – *i.e.* the nature and complexity of the glassy phase, the existence of a transition in nonzero magnetic field, temperature-chaos, etc. – are still debated. Consensus has been reached only for mean-field theory, first derived considering an infinite number of fully-connected Ising spins [3]; its solution by Parisi explicitly breaks the replica symmetry, resulting in a picture where the glassy phase can be decomposed into a set of infinite number and ultrametrically organised pure states [1]. Despite all efforts made to go beyond mean-field theory [4], which is certainly valid in infinite dimensions, finite-dimensional short-ranged systems are much less understood. Beside the mean-field picture, an alternative description, the so-called "droplet picture" has emerged in a series of papers (a list of them, which is certainly not fully complete, is provided as Ref. [5]). This approach claims that replica symmetry breaking is an artifact of mean-field theory, and the glassy phase consists of only two pure phases related by a global inversion of the spins. The droplet theory has gained some support from the field of mathematical physics [6], the conclusions, however, remain disputed [7–9].

A large amount of numerical work[1] has been carried out to resolve the problem, finite-size effects and long relaxation times, however, make it difficult to reach a definite conclusion. It is clear that analytical methods – especially field-theory, as the most powerful of them – are very important to provide reliable results to settle this controversy. A direct field-theoretical study of the glassy phase, however, has proved to be very hard, due to the complexity of the Gaussian propagators and the ubiquity of infrared divergences (see [4] and references therein). A scaling theory for the spin glass phase (just below T_c and in zero magnetic field) and a proposal to handle the infrared problems were put forward in reference [10], still progress in that direction is very slow.

There is one characteristic of the phase diagram in the mean-field picture which is definitely absent in a droplet-like approach, namely the existence of a spin glass tran-

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[1] An extensive list of references for numerical simulations in spin glasses can be found in [8].

sition in a uniform magnetic field [11], known as the de Almeida-Thouless (AT) transition. This question can be studied in the *symmetric* phase, approaching the presumed transition from the high-temperature side, in this way eliminating the problems arising from the complexity of the glassy phase. In the language of replica field-theory, this will lead to a replica symmetric Lagrangean which is invariant under any permutation of the n replicas.

The prime purpose of this paper is to provide the generic field-theoretical model appropriate to a detailed study of the problem raised above, *i.e.* the existence of an AT transition. (In a separate publication [12], the crossover region around the zero-field critical point is elaborated, and the role that a small magnetic field plays in driving the AT transition is investigated). Here we rediscover, at the mean-field level, the importance of the replica number n in the analysis of the AT transition [13,14]: for n small but nonvanishing, the AT transition line moves away from the zero-field critical point, and an intermediate range of temperature emerges even in zero magnetic field. This phase – which can be called, by the extension of the concept of Sherrington [15] to continuous n, the replica magnet phase – is replica symmetric but still has a lower symmetry than the paramagnetic phase. Hence we have two transitions, the first one, in zero field, is an isolated critical point between the two replica symmetric phases (paramagnet and replica magnet), whereas the second one is a whole line in H-T plane between replica magnet and the replica symmetry broken phase. As a result, we can identify the AT transition as the onset of instability of the replica magnet phase, and since it has a lower symmetry than the paramagnet, we must use a generic replica symmetric Lagrangean to study it by field-theoretical methods[2].

From equations (1, 2), one can immediately realize that a perturbational calculation based on that Lagrangean is extremely difficult. This is due to the complicated interaction term with eight different couplings corresponding to the eight possible cubic invariants, and also to the non-diagonal Gaussian-propagators with three distinct bare masses. To overcome these difficulties, we introduce a new representation of the cubic interaction which is associated with a block-diagonalization of the quadratic part. The technique proves to be very efficient, as is displayed in our example where the one-loop calculation of the mass operator is presented[3].

The outline of the paper is as follows: In Section 2 the generic cubic Lagrangean for the replica symmetric field-theory is set forth, first using only symmetry considerations. It is then *derived*, starting from the lattice system, and using the standard Hubbard-Stratonovich transformation. Section 3 is devoted to an analysis of the zero-loop, *i.e.* mean-field, results. The zero-field transition, first discovered by Sherrington and Kirkpatrick [3], proves to be an isolated singularity of the stationary condition, with the unique mass vanishing at that point. On the other hand, one of the three modes becomes massless along the AT surface, signalling the onset of instability of the generic replica symmetric phase. In Section 4 we define the new set of cubic couplings. The introduction of this new representation makes it possible to compute Feynmann-graphs of a perturbative approach; this is illustrated in a one-loop calculation of the three masses in Section 5. A simple and convenient non-orthogonal basis is presented in Appendix A, whereas a detailed list of vertices computed in this basis is given in Appendix B. This almost complete table of vertex rules is published here for later references, an application for an extended renormalisation group study of the finite-dimensional AT transition is in progress [18].

2 Cubic replica field-theory for the Ising spin glass in nonzero magnetic field

After the invention of the renormalization group [19], field theoretical representations of statistical models, originally defined on a lattice, became a standard way to study the behaviour of the systems near phase transitions. The renormalisation group made it possible to use a perturbative method, the loop expansion, in low enough dimensions, thus providing excellent analytical tools to compute phase diagrams and critical properties. The extension to spin glasses came immediately after the replica approach had been introduced by Edwards and Anderson [2], transforming the originally inhomogeneous system into a homogeneous one. A Ginzburg-Landau-Wilson continuum model was first proposed 25 years ago [20,21], then further investigated by renormalisation methods [22,23]. Its cubic Lagrangean was derived – for the Ising case – by Bray and Moore [24] *via* the Hubbard-Stratonovich transformation. Two of the present authors applied the same field theoretical model in their effort to go beyond the mean-field results, and understand the glassy phase of the finite-dimensional short range Ising spin glass [4,10]. The magnetic field was always zero in the above works, with the only exception of [21], where it was introduced by a coupling to the magnetization, leaving the Lagrangean unchanged for the part relevant to the spin glass transition.

Field-theoretical models can be constructed by means of symmetry arguments, building up the Lagrangean from all the possible invariants of the relevant symmetry group of the system. For an Ising spin glass, the fields depend on a pair of replicas, $\phi^{\alpha\beta} = \phi^{\beta\alpha}$ with $\phi^{\alpha\alpha} = 0$, and, as a consequence of the replica trick, any permutation of the n replicas leaves the Lagrangean unchanged. Discarding all the terms higher in the order of the ϕ's than cubic, we arrive at the following *generic replica symmetric* Lagrangean after a search of all the quadratic and cubic invariants:

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(3)},$$

[2] We keep $n \gtrsim 0$ small but finite almost everywhere throughout the paper. This is because we want to present formulae for later calculations in the generic cubic model. For this, however, the $n \to 0$ limit is rather tricky, due to the degeneracy of the longitudinal and anomalous modes at zero n.

[3] Other methods are also available, like Replica Fourier Transform [16], or the usage of projections to the subspaces of the fundamental modes [12,17].

where

$$\mathcal{L}^{(2)} = \frac{1}{2}\sum_{\mathbf{p}}\Bigg[\Big(\frac{1}{2}(pa\rho)^2 + m_1\Big)\sum_{\alpha\beta}\phi^{\alpha\beta}_{\mathbf{p}}\phi^{\alpha\beta}_{-\mathbf{p}} + m_2\sum_{\alpha\beta\gamma}\phi^{\alpha\gamma}_{\mathbf{p}}\phi^{\beta\gamma}_{-\mathbf{p}} + m_3\sum_{\alpha\beta\gamma\delta}\phi^{\alpha\beta}_{\mathbf{p}}\phi^{\gamma\delta}_{-\mathbf{p}}\Bigg], \quad (1)$$

and

$$\mathcal{L}^{(3)} = -\frac{1}{6\sqrt{N}}{\sum_{\mathbf{p_1p_2p_3}}}'\Bigg[w_1\sum_{\alpha\beta\gamma}\phi^{\alpha\beta}_{\mathbf{p}_1}\phi^{\beta\gamma}_{\mathbf{p}_2}\phi^{\gamma\alpha}_{\mathbf{p}_3} + w_2\sum_{\alpha\beta}\phi^{\alpha\beta}_{\mathbf{p}_1}\phi^{\alpha\beta}_{\mathbf{p}_2}\phi^{\alpha\beta}_{\mathbf{p}_3} + w_3\sum_{\alpha\beta\gamma}\phi^{\alpha\beta}_{\mathbf{p}_1}\phi^{\alpha\beta}_{\mathbf{p}_2}\phi^{\alpha\gamma}_{\mathbf{p}_3} + w_4\sum_{\alpha\beta\gamma\delta}\phi^{\alpha\beta}_{\mathbf{p}_1}\phi^{\alpha\beta}_{\mathbf{p}_2}\phi^{\gamma\delta}_{\mathbf{p}_3} + w_5\sum_{\alpha\beta\gamma\delta}\phi^{\alpha\beta}_{\mathbf{p}_1}\phi^{\alpha\gamma}_{\mathbf{p}_2}\phi^{\beta\delta}_{\mathbf{p}_3} + w_6\sum_{\alpha\beta\gamma\delta}\phi^{\alpha\beta}_{\mathbf{p}_1}\phi^{\alpha\gamma}_{\mathbf{p}_2}\phi^{\alpha\delta}_{\mathbf{p}_3} + w_7\sum_{\alpha\beta\gamma\delta\mu}\phi^{\alpha\gamma}_{\mathbf{p}_1}\phi^{\beta\gamma}_{\mathbf{p}_2}\phi^{\delta\mu}_{\mathbf{p}_3} + w_8\sum_{\alpha\beta\gamma\delta\mu\nu}\phi^{\alpha\beta}_{\mathbf{p}_1}\phi^{\gamma\delta}_{\mathbf{p}_2}\phi^{\mu\nu}_{\mathbf{p}_3}\Bigg]. \quad (2)$$

Momentum summations in the above formulae are over the reciprocal vectors of a d-dimensional hypercubic lattice with lattice spacing a, consisting of infinitely many sites N in the thermodynamic limit. The prime in equation (2) means the constraint of momentum conservation, $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$. Neglecting the fluctuations of fields with wavelength much smaller than the range ρa of the exchange interaction between the spins, we confine the relevant values of momentums in equations (1, 2) around the center of the Brillouin zone: $p < \Lambda/\rho a$. The momentum cutoff $\Lambda \ll 1$ makes it possible to expand the nonlocal quadratic coupling in $\mathcal{L}^{(2)}$, and, as it is common in field-theoretical studies of phase transitions, we stop after the first two terms. (The coupling constant of the kinetic term in equation (1) can be set equal to $\frac{1}{2}$ without loss of generality. See also later.)[4]

In zero magnetic field, and in the high-temperature phase where the spin glass order parameter is zero, all the couplings but m_1 and w_1 disappear. In this section, we want to find out the order parameter dependence of the couplings defining our Lagrangean. We are especially interested in the general form of the replica field-theory suitable for studying the de Almeida-Thouless transition in finite dimensions. Our starting point is a standard Edwards-Anderson-like [2] model for N Ising spins on a d-dimensional hypercubic lattice, with a long but finite-ranged interaction:

$$\mathcal{H} = -\sum_{(ij)}\frac{J_{ij}}{\sqrt{z}}f_{ij}s_is_j - H\sum_i s_i. \quad (3)$$

The notation $f_{ij} \equiv f\left(\frac{|\mathbf{r}_i-\mathbf{r}_j|}{\rho a}\right)$ was introduced in the above equation, with the smooth positive function $f(x)$ which takes the value 1 for $x \lesssim 1$, and decays to zero sufficiently fast for $x > 1$, thereby cutting off the interaction around $|\mathbf{r}_i - \mathbf{r}_j| \sim \rho a$. Here $z = \rho^d$ is effectively the coordination number, *i.e.* the number of spins within the interaction radius; expanding quantities in terms of its negative powers will generate the loop-expansion in the replica field-theory. J_{ij} are independent, Gaussian distributed random variables with mean zero and variance Δ^2, and a homogeneous magnetic field H was also included. Summations are over the pairs (ij) of lattice sites in the first sum, while over the N lattice sites in the second one.

In the spirit of the replica trick, we want to compute quantities like the averaged replicated partition function $\overline{Z^n}$ for some positive integer n, finally deducing spin glass behaviour from the $n \to 0$ continuum limit. Averaging first over the quenched disorder results in an effective replica Hamiltonian depending on the spins S_i^α, $\alpha = 1, \ldots, n$[5]:

$$\overline{Z^n} \sim \operatorname*{Tr}_{\{S_i^\alpha\}}\exp\left(\frac{1}{2}\sum_{(\alpha\beta)}\sum_{ij}S_i^\alpha S_i^\beta K_{ij}S_j^\alpha S_j^\beta + \frac{H}{kT}\sum_\alpha\sum_i S_i^\alpha\right),$$

$K_{ij} \equiv \frac{1}{z}\left(\frac{\Delta}{kT}\right)^2 f_{ij}^2$. A Hubbard-Stratonovich transformation can help us to get rid of the four-spin interaction term; the price we have to pay for that is the introduction of integrals over the "fields" $Q_i^{\alpha\beta}$:

$$\overline{Z^n} \sim \Bigg[\prod_{(\alpha\beta),i}\int \mathrm{d}Q_i^{\alpha\beta}\Bigg] \times \exp\left(-\frac{1}{2}\sum_{(\alpha\beta)}\mathbf{Q}^{\alpha\beta}\mathbf{K}^{-1}\mathbf{Q}^{\alpha\beta} + \sum_i \ln\zeta_i\right). \quad (4)$$

The boldfaced vector and matrix notations in equation (4) for $\mathbf{Q}^{\alpha\beta}$ and $\mathbf{K}^{-1}$, respectively, are to simplify the nonlocal term in the formula, whereas the one-spin partition function is defined as follows:

$$\zeta_i = \operatorname*{Tr}_{\{S^\alpha\}}\exp\left(\sum_{(\alpha\beta)}Q_i^{\alpha\beta}S^\alpha S^\beta + \frac{H}{kT}\sum_\alpha S^\alpha\right). \quad (5)$$

To construct a field-theory appropriate for a perturbation expansion around the mean-field solution, *i.e.* around the infinite range model $\rho \to \infty$, we separate $Q_i^{\alpha\beta}$ into its homogeneous, non-fluctuating (mean-field) part, and into its fluctuating part:

$$Q_i^{\alpha\beta} = Q^{\alpha\beta} + \phi_i^{\alpha\beta}. \quad (6)$$

[4] A replica symmetric treatment of the *ordered* phase was carried out in references [23,24]. In this case, the finite order parameter gives rise to a quadratic Lagrangean, even in zero field, which is a special case of $\mathcal{L}^{(2)}$.

[5] Throughout the paper, we use different notations for summations over distinct pairs, $\sum_{(\alpha\beta)} = \sum_{\alpha<\beta}$, and unrestricted sums, $\sum_{\alpha\beta} = \sum_\alpha\sum_\beta$.

When expressed in terms of the ϕ's, the exponent in equation (4) will be called $-\mathcal{L}$, and it can be expanded up to any desired order. Turning to a more convenient representation of the fields in momentum space, contributions up to cubic order have the following forms:

$$\mathcal{L}^{(0)} = N\left[\frac{1}{2}\Theta^{-1}\sum_{(\alpha\beta)}\left(Q^{\alpha\beta}\right)^2 - \ln\zeta\right], \tag{7}$$

$$\mathcal{L}^{(1)} = \sqrt{N}\sum_{(\alpha\beta)}\left[\Theta^{-1}Q^{\alpha\beta} - \langle\langle S^\alpha S^\beta\rangle\rangle\right]\phi^{\alpha\beta}_{\mathbf{p}=0}\,, \tag{8}$$

$$\mathcal{L}^{(2)} = \frac{1}{2}\sum_{\mathbf{p}}\sum_{(\alpha\beta),(\gamma\delta)}\phi^{\alpha\beta}_{\mathbf{p}}M_{\alpha\beta,\gamma\delta}(pa\rho)\phi^{\gamma\delta}_{-\mathbf{p}}, \tag{9}$$

and finally

$$\mathcal{L}^{(3)} = -\frac{1}{6\sqrt{N}}{\sum_{\mathbf{p_1}\mathbf{p_2}\mathbf{p_3}}}'\sum_{(\alpha\beta),(\gamma\delta),(\mu\nu)}W_{\alpha\beta,\gamma\delta,\mu\nu}\phi^{\alpha\beta}_{\mathbf{p}_1}\phi^{\gamma\delta}_{\mathbf{p}_2}\phi^{\mu\nu}_{\mathbf{p}_3}. \tag{10}$$

A Boltzmann-weight with $Q^{\alpha\beta}$, instead of $Q_i^{\alpha\beta}$, is understood in the definitions of ζ and the one-site effective average $\langle\langle\ldots\rangle\rangle$ in equations (7, 8). Θ^{-1} is essentially the temperature squared, or more precisely:

$$\Theta = \left(\frac{\Delta}{kT}\right)^2\int f(r)^2\,\mathrm{d}^d r. \tag{11}$$

The momentum-*dependent* mass, and the momentum-*independent* cubic coupling operators are defined as follows:

$$\begin{aligned}M_{\alpha\beta,\gamma\delta}(pa\rho) = {} & K_{\mathbf{p}}^{-1}\delta^{\mathrm{Kr}}_{\alpha\beta,\gamma\delta} - \big[\langle\langle S^\alpha S^\beta S^\gamma S^\delta\rangle\rangle \\ & - \langle\langle S^\alpha S^\beta\rangle\rangle\langle\langle S^\gamma S^\delta\rangle\rangle\big],\end{aligned} \tag{12}$$

$$\begin{aligned}W_{\alpha\beta,\gamma\delta,\mu\nu} = {} & \langle\langle S^\alpha S^\beta S^\gamma S^\delta S^\mu S^\nu\rangle\rangle \\ & - \langle\langle S^\alpha S^\beta\rangle\rangle\langle\langle S^\gamma S^\delta S^\mu S^\nu\rangle\rangle \\ & - \langle\langle S^\gamma S^\delta\rangle\rangle\langle\langle S^\alpha S^\beta S^\mu S^\nu\rangle\rangle \\ & - \langle\langle S^\mu S^\nu\rangle\rangle\langle\langle S^\alpha S^\beta S^\gamma S^\delta\rangle\rangle \\ & + 2\langle\langle S^\alpha S^\beta\rangle\rangle\langle\langle S^\gamma S^\delta\rangle\rangle\langle\langle S^\mu S^\nu\rangle\rangle\,.\end{aligned} \tag{13}$$

The Kronecker delta in equation (12) represents the $n(n-1)/2$-dimensional unit matrix, whose prefactor comes from the Fourier-transform of K_{ij}:

$$\begin{aligned}K_{\mathbf{p}} &= \frac{1}{N}\sum_{ij}\mathrm{e}^{\mathrm{i}\mathbf{p}(\mathbf{r}_i-\mathbf{r}_j)}K_{ij} \\ &= \frac{1}{z}\left(\frac{\Delta}{kT}\right)^2\frac{1}{N}\sum_{ij}\mathrm{e}^{\mathrm{i}\mathbf{p}(\mathbf{r}_i-\mathbf{r}_j)}f\left(\frac{|\mathbf{r}_i-\mathbf{r}_j|}{\rho a}\right)^2 \\ &\longrightarrow \left(\frac{\Delta}{kT}\right)^2\int\mathrm{e}^{\mathrm{i}(\mathbf{p}a\rho)\mathbf{r}}f(r)^2\,\mathrm{d}^d r.\end{aligned} \tag{14}$$

The arrow in the above formula means the double limiting procedure of the thermodynamic limit ($N\to\infty$), and continuum limit ($a\to 0$). The theory resulting then is a field-theory with all the thermodynamic functions scaling correctly with N, and the lattice constant a disappearing from the momentum integrals after introducing the new variable $\tilde{\mathbf{p}}\equiv\mathbf{p}a\rho$, with the upper momentum-cutoff becoming Λ in $\tilde{\mathbf{p}}$ space. The range ρ of the original interaction, however, survives: a perturbative loop-expansion can be generated where every loop in a Feynmann-diagram contributes a $z^{-1}=\rho^{-d}$ factor.

It is rather common in field-theoretical studies to normalize the fields such that the kinetic term in the Gaussian part of the Lagrangean (that proportional to the squared momentum) be exactly $\tilde{\mathbf{p}}^2$ times the unit matrix. This can be simply reached after expanding $K_{\mathbf{p}}^{-1}$, equation (14), and introducing the new fields by

$$c\,\phi_{\mathbf{p}}\to\phi_{\mathbf{p}}, \tag{15}$$

where

$$c = (2d)^{-\frac{1}{2}}\,\frac{\left(\int r^2 f(r)^2\,\mathrm{d}^d r\right)^{\frac{1}{2}}}{\int f(r)^2\,\mathrm{d}^d r}\,\left(\frac{kT}{\Delta}\right). \tag{16}$$

A corresponding redefinition of the mass operator and cubic interaction,

$$\frac{1}{c^2}\,M\to M \quad\text{and}\quad \frac{1}{c^3}\,W\to W, \tag{17}$$

leaves the form of equations (9, 10) unchanged. Neglecting short-wavelength fluctuations, the mass-operator in equation (12) can be expanded for $\tilde{p}\ll 1$. The commonly used truncation at the kinetic term provides:

$$\begin{aligned}M_{\alpha\beta,\gamma\delta}(\tilde{\mathbf{p}}) = {} & C(d)\,\big[\delta^{\mathrm{Kr}}_{\alpha\beta,\gamma\delta} - \Theta\big(\langle\langle S^\alpha S^\beta S^\gamma S^\delta\rangle\rangle \\ & -\langle\langle S^\alpha S^\beta\rangle\rangle\langle\langle S^\gamma S^\delta\rangle\rangle\big)\big] + \tilde{\mathbf{p}}^2\,\delta^{\mathrm{Kr}}_{\alpha\beta,\gamma\delta},\end{aligned} \tag{18}$$

where $C(d) = 2d\int f(r)^2\,\mathrm{d}^d r/\int r^2 f(r)^2\,\mathrm{d}^d r$ is a smooth function of dimensionality, but independent of the temperature and magnetic field. As such, its concrete value is irrelevant, and a simple adjustment of the cutoff function $f(r)$ can make it equal to unity.

A replica symmetric field-theory – for the study of the massive high-temperature phase, and/or the massless critical manifolds – can be obtained by choosing a replica symmetric mean-field value $Q_{\alpha\beta}\equiv Q$ in equation (6). The stationarity condition $\mathcal{L}^{(1)}\equiv 0$ gives us an implicit equation for Q (see Eq. (8)):

$$\begin{aligned}\Theta^{-1}Q &= \langle\langle S^\alpha S^\beta\rangle\rangle \\ &= \frac{\operatorname*{Tr}_{\{S^\alpha\}}\left(S^\alpha S^\beta\mathrm{e}^{\sum_{(\alpha\beta)}Q\,S^\alpha S^\beta+\frac{H}{kT}\sum_\alpha S^\alpha}\right)}{\operatorname*{Tr}_{\{S^\alpha\}}\left(\mathrm{e}^{\sum_{(\alpha\beta)}Q\,S^\alpha S^\beta+\frac{H}{kT}\sum_\alpha S^\alpha}\right)}\,, \quad \alpha\neq\beta.\end{aligned} \tag{19}$$

Q enters the definition of the mass operator, equation (18), and the cubic interaction operator, equation (13), through the Boltzmann-weight in the averages $\langle\langle\ldots\rangle\rangle$. Replica

Table 1. The relationship between the cubic couplings w's of equation (2) and W's of equation (10).

$$
\begin{aligned}
w_1 &= W_1 - 3W_5 + 3W_7 - W_8 \\
w_2 &= \frac{1}{2}W_2 - 3W_3 + \frac{3}{2}W_4 + 3W_5 + 2W_6 - 6W_7 + 2W_8 \\
w_3 &= 3W_3 - 3W_4 - 6W_5 - 3W_6 + 15W_7 - 6W_8 \\
w_4 &= \frac{3}{4}W_4 - \frac{3}{2}W_7 + \frac{3}{4}W_8 \\
w_5 &= 3W_5 - 6W_7 + 3W_8 \\
w_6 &= W_6 - 3W_7 + 2W_8 \\
w_7 &= \frac{3}{2}W_7 - \frac{3}{2}W_8 \\
w_8 &= \frac{1}{8}W_8
\end{aligned}
$$

symmetry is induced also for these operators, resulting in the three different components of the mass:

$$
\begin{aligned}
M_{\alpha\beta,\alpha\beta}(\tilde{\mathbf{p}}) &= M_1 + \tilde{\mathbf{p}}^2, \\
M_{\alpha\gamma,\beta\gamma}(\tilde{\mathbf{p}}) &= M_2\,, \\
M_{\alpha\beta,\gamma\delta}(\tilde{\mathbf{p}}) &= M_3\,;
\end{aligned}
\tag{20}
$$

and the eight different components of the cubic interaction operator:

$$
\begin{aligned}
W_{\alpha\beta,\beta\gamma,\gamma\alpha} &= W_1\,, \qquad & W_{\alpha\beta,\alpha\beta,\alpha\beta} &= W_2\,, \\
W_{\alpha\beta,\alpha\beta,\alpha\gamma} &= W_3\,, \qquad & W_{\alpha\beta,\alpha\beta,\gamma\delta} &= W_4\,, \\
W_{\alpha\beta,\alpha\gamma,\beta\delta} &= W_5\,, \qquad & W_{\alpha\beta,\alpha\gamma,\alpha\delta} &= W_6\,, \\
W_{\alpha\gamma,\beta\gamma,\delta\mu} &= W_7\,, \qquad & W_{\alpha\beta,\gamma\delta,\mu\nu} &= W_8\,.
\end{aligned}
\tag{21}
$$

$\mathcal{L}^{(2)}$ of equation (9), together with equation (20), is equivalent with that of equation (1), provided the two sets of masses are related by the following expressions:

$$m_1 = \frac{1}{2}(M_1 - 2M_2 + M_3)\,, \tag{22}$$

$$m_2 = M_2 - M_3\,, \tag{23}$$

$$m_3 = \frac{1}{4}M_3\,. \tag{24}$$

Similarly, equations (2, 10) are two different representations of $\mathcal{L}^{(3)}$. Using equation (21), a one to one correspondence between the two sets of cubic couplings, w's and W's, can be deduced by a somewhat lengthy but elementary calculation. The results are listed in Table 1.

3 Analysis of the stationarity conditions and bare masses

It is easy to recognize that, after a simple rescaling of the temperature, equation (19) coincides with the replica symmetric mean-field equation of Sherrington and Kirkpatrick (SK) [3,25] for the order parameter of the Ising spin glass on a fully-connected lattice. This may not be surprising: the most direct way to define mean-field theory on a d-dimensional hypercubic lattice is letting ρ, the range of interaction, go to infinity, thus neglecting all the loop corrections to the equation of state [26,27]. The solution Q of equation (19) has, however, an application that goes beyond mean-field theory: it enters the mass operator and interaction components in the formulae equations (18, 13), respectively, through the effective average $\langle\langle \dots \rangle\rangle$. Although a field-theory emerging from this procedure has a direct connection to the original parameters, such as temperature, magnetic field and also replica number n, renormalisation will reshuffle the masses and couplings, possibly forcing them into some fixed point.

Following references [3,25], equation (19) can be cast into a more convenient form:

$$
\begin{aligned}
&\Theta^{-1}Q = \\
&\frac{\int \frac{\mathrm{d}u}{\sqrt{2\pi}} \mathrm{e}^{-\frac{u^2}{2}} \tanh^2(\sqrt{Q}u + H/kT) \cosh^n(\sqrt{Q}u + H/kT)}{\int \frac{\mathrm{d}u}{\sqrt{2\pi}} \mathrm{e}^{-\frac{u^2}{2}} \cosh^n(\sqrt{Q}u + H/kT)} \\
&\qquad = \overline{\tanh^2(\sqrt{Q}u + H/kT)}.
\end{aligned}
\tag{25}
$$

The shorthand notation

$$
\begin{aligned}
&\overline{\tanh^k(\dots)} \equiv \\
&\qquad \frac{\int \frac{\mathrm{d}u}{\sqrt{2\pi}} \mathrm{e}^{-\frac{u^2}{2}} \tanh^k(\dots) \cosh^n(\sqrt{Q}u + H/kT)}{\int \frac{\mathrm{d}u}{\sqrt{2\pi}} \mathrm{e}^{-\frac{u^2}{2}} \cosh^n\left(\sqrt{Q}u + H/kT\right)}
\end{aligned}
\tag{26}
$$

has been introduced for later use. By equation (25), Q is implicitly given as a function of temperature, magnetic field and replica number. One can easily find the SK spin glass transition point ($\Theta = 1$, $H = 0$) as an isolated singularity for *any* given n close to zero. (Keeping n finite is for later use. At the moment, we must notice that this singularity is rather unaffected by the $n \to 0$ limit.) The relevant, positive, solution for $H = 0$ is

$$
Q = \begin{cases} 0 & \text{for } t > 0, \\ -\frac{1}{2-n}t + \left[-\frac{n-3}{(n-2)^2} + \frac{1}{3(n-2)^3}\right] t^2 + \dots & \text{for } t < 0; \end{cases}
\tag{27}
$$

where the new temperature scale $t \equiv \Theta^{-1} - 1 > 0$ (< 0) in the disordered (spin glass) phase, respectively. As displayed in Figure 1, one can join up smoothly the two regimes by-passing the critical point, like in ordinary critical phenomena[6].

[6] It is obvious from equation (25) that $Q = 0$ is always a solution for zero magnetic field, independently of t and n. The $Q = 0$ solution for $t < 0$, however, defines the negative Q branch (starting from point G in Figure 1a with $Q = 0$, and following the dotted curve, you end up at P with $Q < 0$) which is non-physical. The two branches meet at the SK transition point.

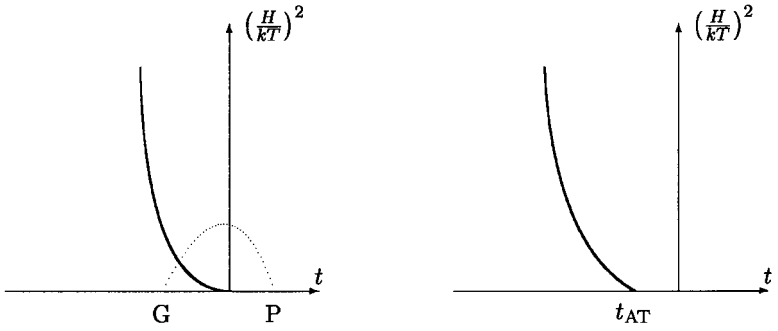


Fig. 1. (a): Mean-field phase diagram for $n = 0$. Starting from the paramagnetic state P with $Q = 0$, and following the dotted curve, the SK critical point ($t = H = 0$) can be by-passed ending in the glassy state G with $Q > 0$. (b): Mean-field phase diagram for $n \gtrsim 0$. There is a temperature range even in zero field, $t_{AT} < t < 0$, where Q is positive and the replica symmetric state is stable.

To go beyond a mean-field solution, and build loops, one must have well-defined Gaussian propagators, *i.e.* the eigenvalues of the mass operator must be non-negative. A generic replica symmetric mass operator, like that in equation (20), was diagonalised years ago [23,24] with the following expressions for the eigenvalues (which will play the role of bare masses here):

$$r_R = M_1 - 2M_2 + M_3, \tag{28}$$

$$r_A = M_1 + (n-4)M_2 - (n-3)M_3, \tag{29}$$

$$r_L = M_1 + 2(n-2)M_2 + \frac{(n-2)(n-3)}{2}M_3. \tag{30}$$

The indices R, A and L stand for replicon, anomalous and longitudinal, respectively, each referring to the corresponding family of eigenmodes (see Ref. [24] and also later sections). From equations (18, 20), and using the stationarity condition (19), the following expressions are obtained for the bare masses:

$$r_R = \frac{t}{1+t} + 2Q - \frac{1}{1+t}\,\overline{\tanh^4(\sqrt{Q}u + H/kT)}, \tag{31}$$

$$r_A = r_R + (n-2)\left[\frac{1}{1+t}\,\overline{\tanh^4(\sqrt{Q}u + H/kT)} - Q\right], \tag{32}$$

$$r_L = r_R + \frac{n-1}{2}\Bigg[-(n-4)\frac{1}{1+t}\,\overline{\tanh^4(\sqrt{Q}u + H/kT)} - 4Q + n(1+t)Q^2\Bigg]. \tag{33}$$

An expansion below the transition provides, in zero field,

$$r_R = \frac{n}{2}(1 + \frac{n}{2} + \dots)(-t) + (-\frac{1}{3} + \dots)\,t^2 + \dots, \tag{34}$$

$$r_A = (1 + \frac{n}{2} + \dots)(-t) + (-\frac{5}{12} + \dots)\,t^2 + \dots, \tag{35}$$

$$r_L = (-t) + (-\frac{5}{12} + \dots)\,t^2 + \dots. \tag{36}$$

(The n-dependent coefficients are displayed in an expanded form for showing clearly the signs for small n. The complete n-dependence, however, is easily found.) The hitherto degenerate masses split when passing the SK transition singularity (where they are zero) and emerge positively in the spin glass phase for any $n \gtrsim 0$. r_R starts, however, with a small slope proportional to n, and becomes massless again at the AT surface $t_{AT} = -\frac{3}{2}n + \dots$, where instability of the replica symmetric phase begins. This result was first derived by Kondor [13], the aspect we wish to emphasize here is the existence of an intermediary temperature range where replica symmetry persists, see Figure 1, though, as a result of a nonzero Q, the level of symmetry is lower, leading, at the mean-field level, to the splitting of the bare masses.

By definition, $r_R \equiv 0$ on the AT-surface. From equations (25, 31), the magnetic field can be expressed as a double series for small $-t$ and n; the leading, cubic, term is as follows:

$$(H/kT)^2 = \frac{1}{6}(-t)^3 - \frac{1}{4}nt^2 = -\frac{3}{8}n^2(t - t_{AT}) + \frac{1}{2}n(t - t_{AT})^2 - \frac{1}{6}(t - t_{AT})^3.$$

This can be cast into a scaling form

$$(H/kT)^2 = t^\tau \varphi(\frac{n}{t^\kappa}), \qquad t, n \to 0, \tag{37}$$

where the exponents have now their mean-field values $\tau = 3$ and $\kappa = 1$, and $\varphi(\dots)$ is the scaling function characterising the AT-surface.

On the basis of the above mean-field analysis, we can define two different, though both replica symmetric, cubic field-theories relevant to describe spin glass transitions of different types in low enough dimensions (d certainly smaller than eight):

– The zero-field model – $H \equiv 0, t \geq 0$ and $n \gtrsim 0$ – implies, through equation (27), $Q \equiv 0$, leading to degenerate bare masses $r_R = r_A = r_L$, see equations (31, 32) and (33). It is easy to verify using equations (13, 21) that all the W's but W_1 are zero, and the same is true, by Table 1, for the small w's. Let us put down

explicitly the definition of the cubic field theory for the zero-field transition from the paramagnetic phase:

$$\begin{aligned} r \equiv & r_{\mathrm{R}} = r_{\mathrm{A}} = r_L; \\ w \equiv & w_1, \qquad w_i = 0, \quad i = 2, \ldots, 8. \end{aligned} \tag{38}$$

At the mean-field level, a massless state is reached along the critical *line* $H = 0$, $t = 0$ and $n \gtrsim 0$; the replica number being a rather innocent parameter around zero. We expect this property to remain for short-ranged (finite-dimensional) systems, and indeed, ϵ-expansion results have supported this idea [28]. (This kind of replica field-theory was studied in all the references [20–24,28].)

– The second model, which is in fact the most general cubic field-theory with an unbroken replica symmetry, has all the three masses m_i and eight couplings w_i different. At the mean-field level, it corresponds to a nonzero Q which is always such when a magnetic field is switched on. More surprisingly, however, there is a whole range of temperatures $t_{\mathrm{AT}} \leq t < 0$ even in zero field where the bare parameters correspond to this more general model. Criticality is induced, at least at the mean-field level, by the masslessness of the replicon mode on the de Almeida-Thouless *surface*. The replica number n is now a crucially important parameter; a fact clearly shown by the scaling formula (37). How fluctuations will modify this picture is a prime problem in spin glass theory. There has been two attempts, at least to our knowledge, addressing this question [17,29]. In reference [17] fluctuations were restricted to the replicon subspace; in our language this means for the masses that $r_{\mathrm{A}} = r_{\mathrm{L}} = \infty$ and r_{R} critical, while all the cubic couplings were zero but w_1 and w_2. The effect of a small magnetic field was introduced in reference [29] by shifting the bare masses from their zero-field values (more precisely, beside m_1, m_2 became massive too), the couplings remained, however, unchanged. The replica number was effectively set to zero in these works. Here we wish to emphasize the role n may play in a search for a de Almeida-Thouless transition in finite-dimensional systems.

4 A canonical representation of the cubic interaction

In a field-theory with more than one mass, as in the generic replica symmetric system introduced in the previous sections, critical manifolds can be classified by their massless eigenmodes. In this case, it is more convenient to use directly the *eigenvalues* of the mass operator (r_{R}, r_{A} and r_{L}; equations (28, 29) and (30)), instead of the sets m's or M's. At that point it is natural to ask what will happen with the interaction vertices whose legs join the, by this time block-diagonalized, propagators. We show in this section how the transformation that block-diagonalizes the quadratic part of the Lagrangean into "modes" induces a new set of cubic couplings describing how these modes interact. Our replicated field-theory becomes more tractable after using these "canonical" cubic parameters.

The $\frac{1}{2}n(n-1)$-dimensional vectorspace spanned by the two-replica fields $\phi^{\alpha\beta}$ has the simple structure being a direct sum of the subspaces called longitudinal, anomalous and replicon. Their definitions are as follows ($\phi^{\alpha\beta} = \phi^{\beta\alpha}$ and $\phi^{\alpha\alpha} = 0$ are understood everywhere, of course):

– The longitudinal (L) subspace consists of replica symmetric vectors, *i.e.* independent of replica indices. Each element from this subspace corresponds to a *scalar* ϕ:

$$\phi^{\alpha\beta}_{\mathrm{L}} = \phi, \tag{39}$$

and it is obviously one-dimensional.

– Any element of the anomalous (A) subspace can be represented by a one-replica field ϕ^{α}, *i.e.* by a *vector* restricted, however, by the condition

$$\sum_{\alpha} \phi^{\alpha} = 0. \tag{40}$$

A generic anomalous field can now be written as

$$\phi^{\alpha\beta}_{\mathrm{A}} = \frac{1}{2}(\phi^{\alpha} + \phi^{\beta}). \tag{41}$$

As a result of condition (40), the anomalous subspace is $n-1$-dimensional.

– True two-replica fields, loosely speaking *tensors*, constitute the replicon (R) subspace with the restriction

$$\sum_{\beta} \phi^{\alpha\beta}_{\mathrm{R}} = 0 \quad \text{for any} \quad \alpha = 1 \ldots n. \tag{42}$$

From the n equations above follows that the number of independent $\phi^{\alpha\beta}_{\mathrm{R}}$ is $\frac{1}{2}n(n-1) - n = \frac{1}{2}n(n-3)$, rendering the replicon subspace $\frac{1}{2}n(n-3)$-dimensional.

A generic field $\phi^{\alpha\beta}$ can always be decomposed into the sum

$$\phi^{\alpha\beta} = \phi^{\alpha\beta}_{\mathrm{L}} + \phi^{\alpha\beta}_{\mathrm{A}} + \phi^{\alpha\beta}_{\mathrm{R}}. \tag{43}$$

It is straightforward to see that the subspaces defined above give the exact diagonalisation of a *generic* replica symmetric matrix, as defined in the equations of (20), *i.e.*

$$\sum_{(\gamma\delta)} M_{\alpha\beta,\gamma\delta}\phi^{\gamma\delta}_i = r_i \phi^{\alpha\beta}_i, \qquad i = \mathrm{L}, \mathrm{A}, \mathrm{R}, \tag{44}$$

the eigenvalues given in equations (30, 29) and (28), respectively. For a generalization to higher order operators, the matrix-element of M between two arbitrary vectors $\phi^{\alpha\beta}$ and $\psi^{\alpha\beta}$ will be computed after having represented them by the longitudinal scalar, anomalous vector and replicon tensor, as explained above (see Eqs. (39, 41) and (43)). An expression in terms of the three second-order invariants $\phi\psi$, $\sum_{\alpha}\phi^{\alpha}\psi^{\alpha}$ and $\sum_{\alpha\beta}\phi^{\alpha\beta}_{\mathrm{R}}\psi^{\alpha\beta}_{\mathrm{R}}$ arises:

$$\begin{aligned} M_{\{\phi,\psi\}} \equiv & \sum_{(\alpha\beta),(\gamma\delta)} M_{\alpha\beta,\gamma\delta}\phi^{\alpha\beta}\psi^{\gamma\delta} \\ = & \frac{1}{2} r_{\mathrm{R}} \sum_{\alpha\beta} \phi^{\alpha\beta}_{\mathrm{R}}\psi^{\alpha\beta}_{\mathrm{R}} \\ & + \frac{n-2}{4} r_{\mathrm{A}} \sum_{\alpha} \phi^{\alpha}\psi^{\alpha} + \frac{n(n-1)}{2} r_{\mathrm{L}}\phi\psi \end{aligned} \tag{45}$$

(to get the anomalous term, the restriction (40) has been used). Except for numerical factors, the corresponding eigenvalues appear as the coefficients of the three possible second order invariants; *viz.* RR, AA and LL.

The most important point we can learn from equation (45) is a complete factorization of a matrixelement, provided ϕ and ψ are chosen from one of the subspaces L, A or R; *i.e.*

$$M_{\{\phi,\psi\}} = \{\text{expression of mass components}\} \times \{\text{invariant composed of } \phi \text{ and } \psi\}\cdot \quad (46)$$

Generalization to a cubic replica symmetric operator, as defined in equation (21), is straightforward. The analogue of the matrixelement can be easily defined by

$$W_{\{\phi,\psi,\chi\}} \equiv \sum_{(\alpha\beta),(\gamma\delta),(\mu\nu)} W_{\alpha\beta,\gamma\delta,\mu\nu}\phi^{\alpha\beta}\psi^{\gamma\delta}\chi^{\mu\nu}. \quad (47)$$

Taking each of the fields ϕ, ψ and χ from one of the subspaces L, A or R, the nonzero values obtained can be listed as follows:

$$\begin{aligned}
W_{\mathrm{RRR}} &= g_1 \sum_{\alpha\beta\gamma}\phi^{\alpha\beta}\psi^{\beta\gamma}\chi^{\gamma\alpha} + \frac{1}{2}g_2\sum_{\alpha\beta}\phi^{\alpha\beta}\psi^{\alpha\beta}\chi^{\alpha\beta},\\
W_{\mathrm{RRA}} &= g_3 \sum_{\alpha\beta}\phi^{\alpha\beta}\psi^{\alpha\beta}\chi^{\alpha},\\
W_{\mathrm{RRL}} &= g_4 \sum_{\alpha\beta}\phi^{\alpha\beta}\psi^{\alpha\beta}\chi,\\
W_{\mathrm{RAA}} &= g_5 \sum_{\alpha\beta}\phi^{\alpha\beta}\psi^{\alpha}\chi^{\beta},\\
W_{\mathrm{AAA}} &= g_6 \sum_{\alpha}\phi^{\alpha}\psi^{\alpha}\chi^{\alpha},\\
W_{\mathrm{AAL}} &= g_7 \sum_{\alpha}\phi^{\alpha}\psi^{\alpha}\chi,\\
W_{\mathrm{LLL}} &= g_8\,\phi\psi\chi.
\end{aligned} \quad (48)$$

Symmetry makes the remaining W_{RAL}, W_{RLL} and W_{ALL} all identically zero. As for the masses, a complete factorization occurs in the above formula, except the RRR vertex[7]. We prefer the name "canonical" for the set of cubic parameters g emerging in the above formulas, as they are, in some sense, an extension of the notion of eigenvalues to the cubic interaction term. After a somewhat lengthy calculation, we obtained the set of equations for the g_i in terms of the W_i, $i = 1,\ldots,8$, which are the counterparts of equations (28, 29) and (30). (We omit to display these rather complicated, and not very instructive, expressions here; they can be easily obtained from equations (49a–h) below and using Tab. 1).

Comparing equations (2, 48), a one to one correspondence between a w_i and a g_i, $i = 1,\ldots,8$, is obvious. When expressing the g's in terms of the w's, instead of the W's, not only the formulas become simpler, but a clear R $\to$ A $\to$ L hierarchy emerges:

$$g_1 = w_1, \quad (49a)$$

$$g_2 = 2w_2, \quad (49b)$$

$$g_3 = -w_1 + w_2 + \frac{n-2}{6}w_3, \quad (49c)$$

$$g_4 = -w_1 + w_2 + \frac{n-1}{3}w_3 + \frac{n(n-1)}{3}w_4, \quad (49d)$$

$$g_5 = \frac{n-4}{4}w_1 + \frac{1}{2}w_2 + \frac{n-2}{6}w_3 + \frac{(n-2)^2}{12}w_5, \quad (49e)$$

$$g_6 = -\frac{3n-8}{4}w_1 + \frac{n-4}{4}w_2 + \frac{(n-2)(n-4)}{8}w_3 - \frac{(n-2)^2}{4}w_5 + \frac{(n-2)^3}{8}w_6, \quad (49f)$$

$$g_7 = \frac{n-2}{2}\Big[\frac{n-4}{2}w_1 + w_2 + \frac{2n-3}{3}w_3 + \frac{n(n-1)}{3}w_4 + \frac{(n-2)(2n-3)}{6}w_5 + \frac{(n-1)(n-2)}{2}w_6 + \frac{n(n-1)(n-2)}{6}w_7\Big], \quad (49g)$$

$$g_8 = n(n-1)\Big[(n-2)w_1 + w_2 + (n-1)w_3 + n(n-1)w_4 + (n-1)^2w_5 + (n-1)^2w_6 + n(n-1)^2w_7 + n^2(n-1)^2w_8\Big]. \quad (49h)$$

5 Illustration of the technique: one-loop calculation of the masses

We apply the standard definition of the mass operator, *i.e.* it is the zero momentum limit of the inverse of the two-point function:

$$\Gamma_{\alpha\beta,\gamma\delta} \equiv \lim_{\mathbf{p}\to 0}\big[G^{-1}(\mathbf{p})\big]_{\alpha\beta,\gamma\delta} \quad (50)$$

where the *connected* two-point function G is the average

$$\big[G(\mathbf{p})\big]_{\alpha\beta,\gamma\delta} \equiv \langle\phi^{\alpha\beta}_{\mathbf{p}}\phi^{\gamma\delta}_{-\mathbf{p}}\rangle - \langle\phi^{\alpha\beta}_{\mathbf{p}}\rangle\langle\phi^{\gamma\delta}_{-\mathbf{p}}\rangle \quad (51)$$

taken with the statistical weight $\sim \mathrm{e}^{-\mathcal{L}}$, $\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(3)}$; see equations (1, 2). Dyson's equation for Γ allows us to compute it perturbatively:

$$\Gamma_{\alpha\beta,\gamma\delta} = M_{\alpha\beta,\gamma\delta}(\tilde{\mathbf{p}}=0) - \Sigma_{\alpha\beta,\gamma\delta}(\tilde{\mathbf{p}}=0), \quad (52)$$

where the mean-field mass operator $M_{\alpha\beta,\gamma\delta}$ has been defined in equations (18, 20), whereas the self-energy Σ contains all the one-particle irreducible graphs to the two-point function, with external lines omitted. Up to leading, one-loop, order it is given as the simple "bubble" diagram:

$$\Sigma_{\alpha\beta,\gamma\delta}(\tilde{\mathbf{p}}=0) = \frac{1}{2z}\int^{\Lambda}\frac{\mathrm{d}^d\tilde{p}}{(2\pi)^d}\sum_{\substack{(\alpha'\beta'),(\gamma'\delta')\\(\alpha''\beta'')(\gamma''\delta'')}} W_{\alpha\beta,\alpha'\beta',\alpha''\beta''} \times G^{(0)}_{\alpha'\beta',\gamma'\delta'}(\tilde{p})G^{(0)}_{\alpha''\beta'',\gamma''\delta''}(\tilde{p})W_{\gamma\delta,\gamma'\delta',\gamma''\delta''} \quad (53)$$

[7] The reason for that is the two different cubic invariants we can construct from replicon fields; see the first line of equation (48).

where the free propagator $G^{(0)}$ is defined by

$$G^{(0)}_{\alpha\beta,\gamma\delta}(\tilde{p}) \equiv \left[M^{-1}(\tilde{p})\right]_{\alpha\beta,\gamma\delta}, \tag{54}$$

see equations (18, 20), and the bare vertices, W's, have been introduced in equations (10, 13) and (21).

To compute the replica sum in equation (53), we must overcome the problem of having non-diagonal free propagators. This project can be easily accomplished by calculating $\Sigma_{m,n}$, instead of $\Sigma_{\alpha\beta,\gamma\delta}$, defined as:

$$\Sigma_{m,n} \equiv \sum_{(\alpha\beta),(\gamma\delta)} \phi^{\alpha\beta}_m \, \Sigma_{\alpha\beta,\gamma\delta} \, \phi^{\gamma\delta}_n, \tag{55}$$

ϕ_m and ϕ_n taken from the non-orthogonal basis discussed in Appendix A. Exploiting the completeness of both this basis and its biorthogonal counterpart[8], we can transform $\Sigma_{m,n}$ into a representation with *diagonal* free propagators:

$$G^{(0)}_{m,\tilde{n}} \equiv \sum_{(\alpha\beta),(\gamma\delta)} \phi^{\alpha\beta}_m \, G^{(0)}_{\alpha\beta,\gamma\delta} \, \tilde{\phi}^{\gamma\delta}_n = G^{(0)}_m \, \delta^{\mathrm{Kr}}_{mn} \tag{56}$$

(a "tilde" refers always to a member of the reciprocal basis). $G^{(0)}_m$, as it is the eigenvalue of the free propagator matrix, can be simply related, through equation (54), to one of the three bare masses of equations (28, 29) and (30):

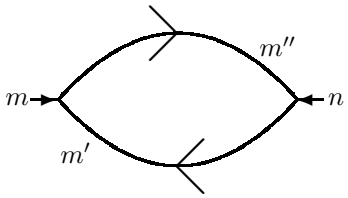
$$G^{(0)}_m(\tilde{p}) = \frac{1}{r_m + \tilde{p}^2}, \tag{57}$$

$r_m = r_{\mathrm{R}}$, r_{A} or r_{L} depending on the subspace m belongs to (see Appendix A).

We can now propose a simple graphical representation for $\Sigma_{m,n}$ by introducing arrowed lines for the free propagators $G^{(0)}$ joining interaction vertices W:

$$G^{(0)}_m \quad \Leftrightarrow \quad \xrightarrow{\qquad m \qquad} \quad .$$

Using the convention of Appendix B concerning the meaning of inward and outward arrows, we can draw for $\Sigma_{m,n}$:



which can be spelled out explicitly as:

$$\Sigma_{m,n} = \frac{1}{2z} \int^{\Lambda} \frac{\mathrm{d}^d\tilde{p}}{(2\pi)^d} \sum_{m',m''} W_{m,m',\tilde{m}''} \, W_{n,m'',\tilde{m}'} \, G^{(0)}_{m'}(\tilde{p}) \, G^{(0)}_{m''}(\tilde{p}). \tag{58}$$

[8] What we use in the derivation of this formula is the decomposition of the unit operator: $\delta^{\mathrm{Kr}}_{\alpha\beta,\gamma\delta} = \sum_m \phi^{\alpha\beta}_m \tilde{\phi}^{\gamma\delta}_m = \sum_m \tilde{\phi}^{\alpha\beta}_m \phi^{\gamma\delta}_m$.

(Summations are over the set of $n(n-1)/2$ modes of Appendix A.) Orthogonality of the different subspaces restricts the number of nonzero elements of the matrix $\Sigma_{m,n}$ to the cases where m and n belong to the same family R, A or L. To compute the three eigenvalues of the self-energy, we can make the simplest possible choices for m and n, *i.e.*

$$\begin{aligned}\Sigma_{(\mu\nu),(\mu\nu)} &= \Sigma_{\mathrm{R}} \sum_{(\alpha\beta)} \phi^{\alpha\beta}_{(\mu\nu)} \phi^{\alpha\beta}_{(\mu\nu)} \\ &= \frac{(n-1)(n-2)^2(n-3)}{4} \, \Sigma_{\mathrm{R}}\,, \end{aligned} \tag{59}$$

$$\begin{aligned}\Sigma_{(\mu),(\mu)} &= \Sigma_{\mathrm{A}} \sum_{(\alpha\beta)} \phi^{\alpha\beta}_{(\mu)} \phi^{\alpha\beta}_{(\mu)} \\ &= \frac{n(n-1)(n-2)}{4} \, \Sigma_{\mathrm{A}}\,, \end{aligned} \tag{60}$$

$$\Sigma_{(L),(L)} = \Sigma_{\mathrm{L}} \sum_{(\alpha\beta)} \phi^{\alpha\beta}_{(\mathrm{L})} \phi^{\alpha\beta}_{(\mathrm{L})} = \frac{n(n-1)}{2} \, \Sigma_{\mathrm{L}}\,. \tag{61}$$

(The computation of the scalar products above is relatively easy using the definitions of the basis functions in Appendix A.) An extensive use of the table of cubic vertices $W_{m,m',\tilde{m}''}$ in Appendix B makes it possible to compute the left-hand sides; the feasibility of the calculation is, however, due to the *selection rule* we explain in that appendix. The results can be summarized by displaying the eigenmodes of the mass operator Γ, by means of equation (52), valid to first order in $1/z$:

$$\begin{aligned}\Gamma_{\mathrm{R}} = r_{\mathrm{R}} - \Bigg\{ \Big[& \frac{n^4 - 8n^3 + 19n^2 - 4n - 16}{(n-1)(n-2)^2} \, g_1^2 \\ & + \frac{2(3n^2 - 15n + 16)}{(n-1)(n-2)^2} \, g_1 g_2 \\ & + \frac{n^3 - 9n^2 + 26n - 22}{2(n-1)(n-2)^2} \, g_2^2 \Big] \, I_{\mathrm{RR}} \\ & + \frac{8(n-1)(n-4)}{n(n-2)^2} \, g_3^2 \, I_{\mathrm{RA}} + \frac{8}{n(n-1)} \, g_4^2 \, I_{\mathrm{RL}} \\ & + \frac{16}{(n-2)^2} \, g_5^2 \, I_{\mathrm{AA}} \Bigg\}; \end{aligned} \tag{62}$$

$$\begin{aligned}\Gamma_{\mathrm{A}} = r_{\mathrm{A}} - \Bigg\{ & \frac{2(n-3)(n-4)}{(n-2)^2} \, g_3^2 \, I_{\mathrm{RR}} \\ & + \frac{16n(n-3)}{(n-1)(n-2)^2} \, g_5^2 \, I_{\mathrm{RA}} + \frac{32}{n(n-2)^2} \, g_6^2 \, I_{\mathrm{AA}} \\ & + \frac{32}{n(n-1)(n-2)^2} \, g_7^2 \, I_{\mathrm{AL}} \Bigg\}; \end{aligned} \tag{63}$$

$$\begin{aligned}\Gamma_{\mathrm{L}} = r_{\mathrm{L}} - \Bigg\{ & \frac{2(n-3)}{n-1} \, g_4^2 \, I_{\mathrm{RR}} + \frac{16}{n(n-2)^2} \, g_7^2 \, I_{\mathrm{AA}} \\ & + \frac{4}{n^3(n-1)^3} \, g_8^2 \, I_{\mathrm{LL}} \Bigg\}. \end{aligned} \tag{64}$$

To help the reader to understand the structure of the corrections to the masses, we introduced the short-cut notation

$$I_{ss'} \equiv \frac{1}{z} \int^{\Lambda} \frac{\mathrm{d}^d\tilde{p}}{(2\pi)^d} \, \frac{1}{r_s + \tilde{p}^2} \, \frac{1}{r_{s'} + \tilde{p}^2}$$

for the momentum integrals; s and s' correspond to one of the subspaces R, A or L. By means of the transformation rules between the g and w couplings, equations (49a–h), the above equations for the masses can be easily expressed in terms of the w's too.

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Appendix A: A simple non-orthogonal basis

For applying the canonical vertices defined in equations (47, 48), it is necessary to introduce a basis in each of the subspaces; an obviously nonunique task. The non-orthogonal basis defined below is not only the simplest[9], but cubic vertices evaluated in this basis will have a remarkable property, a kind of a selection rule involving replica numbers, making computation of Feynmann-graphs feasible (see Appendix B).

A member of this non-orthogonal basis will be denoted by $\phi_m^{\alpha\beta}$, whereas its biorthogonal[10] counterpart, a member of the reciprocal basis, as $\tilde{\phi}_m^{\alpha\beta}$, where m stands for the modes in the subspaces L, A and R as follows:

- The L subspace is one-dimensional, *i.e.* $m \leftrightarrow (L)$.
- m runs the single replica numbers, except one (which we choose the nth), for the $n-1$-dimensional A space: $m \leftrightarrow (\mu)$, $\mu = 1, \ldots, n-1$.
- In case of replicon modes, m corresponds to a pair of replicas, $m \leftrightarrow (\mu\nu)$, with $\mu,\nu = 1, \ldots, n-1$ and $\mu \neq \nu$. To ensure the correct dimensionality $n(n-3)/2$ imposed by condition (42), we have to pick out an arbitrarily chosen pair $(\bar{\mu}\bar{\nu})$, giving for the number of replicon modes:

$$\frac{(n-1)(n-2)}{2} - 1 = \frac{n(n-3)}{2}.$$

To sum up, there are two types of replicon modes:

$$m \leftrightarrow \begin{cases} (\mu\nu), & \mu,\nu = 1, \ldots, n-1; \\ \mu,\nu \neq \bar{\mu} \quad \text{or} \quad \bar{\nu}, & \\ (\mu\bar{\mu}) \quad \text{or} \quad (\mu\bar{\nu}), & \mu = 1, \ldots, n-1; \\ \mu \neq \bar{\mu} \quad \text{or} \quad \bar{\nu}. & \end{cases} \tag{65}$$

In what follows we want to collect the results, omitting any proof.

L subspace:

$$\phi_{(\mathrm{L})}^{\alpha\beta} \equiv \phi_{(\mathrm{L})} = 1; \qquad \tilde{\phi}_{(\mathrm{L})}^{\alpha\beta} \equiv \tilde{\phi}_{(\mathrm{L})} = \frac{2}{n(n-1)}\cdot \tag{66}$$

A subspace:

$$\phi_{(\mu)}^{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha,\beta \neq \mu, \\ -\frac{n-2}{2} & \text{if } \alpha = \mu \text{ or } \beta = \mu; \end{cases}$$

$$\tilde{\phi}_{(\mu)}^{\alpha\beta} = \frac{4}{n^2(n-2)}\Big[\phi_{(\mu)}^{\alpha\beta} + \sum_{\nu=1}^{n-1} \phi_{(\nu)}^{\alpha\beta}\Big]. \tag{67}$$

The one-replica objects, introduced in equation (41), representing them are

$$\phi_{(\mu)}^{\alpha} = \begin{cases} 1 & \text{if } \alpha \neq \mu, \\ -(n-1) & \text{if } \alpha = \mu; \end{cases}$$

$$\tilde{\phi}_{(\mu)}^{\alpha} = \begin{cases} 0 & \text{if } \alpha \neq \mu, n, \\ -\frac{4}{n(n-2)} & \text{if } \alpha = \mu, \\ \frac{4}{n(n-2)} & \text{if } \alpha = n. \end{cases} \tag{68}$$

R subspace:

$$\phi_{(\mu\nu)}^{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha,\beta \neq \mu,\nu\,, \\ -\frac{n-3}{2} & \text{if there is one common replica index of the two pairs } \alpha,\beta \text{ and } \mu,\nu\,, \\ \frac{(n-2)(n-3)}{2} & \text{if the two pairs } \alpha,\beta \text{ and } \mu,\nu \text{ are identical.} \end{cases} \tag{69}$$

In the reciprocal basis, we have different forms for the two types of replicon modes defined in equation (65):

type-I

$$\tilde{\phi}_{(\mu\nu)}^{\alpha\beta} = \begin{cases} \frac{2}{(n-1)(n-2)} & \text{if } (\alpha\beta) = (\mu\nu) \text{ or } (\bar{\mu}n) \text{ or } (\bar{\nu}n), \\ -\frac{2}{(n-1)(n-2)} & \text{if } (\alpha\beta) = (\bar{\mu}\bar{\nu}) \text{ or } (\mu n) \text{ or } (\nu n), \\ 0 & \text{otherwise;} \end{cases} \tag{70}$$

type-II

$$\tilde{\phi}_{(\bar{\mu}\mu)}^{\alpha\beta} = \begin{cases} \frac{2}{(n-1)(n-2)} & \text{if } (\alpha\beta) = (\bar{\mu}\mu) \text{ or } (\bar{\nu}n), \\ -\frac{2}{(n-1)(n-2)} & \text{if } (\alpha\beta) = (\bar{\mu}\bar{\nu}) \text{ or } (\mu n), \\ 0 & \text{otherwise.} \end{cases} \tag{71}$$

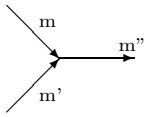
Appendix B: Vertex rules

For a cubic vertex with $\phi = \phi_m$, $\psi = \phi_{m'}$ and $\chi = \tilde{\phi}_{m''}$ in equation (47) (m, m' and m'' referring to the modes introduced in Appendix A), a simple graphical representation

[9] This time, simplicity and orthogonality contradict each other. An orthogonal, still rather complicated system was proposed years ago [30].

[10] Biorthogonality has the usual definition $\sum_{(\alpha\beta)} \phi_m^{\alpha\beta}\tilde{\phi}_{m'}^{\alpha\beta} = \delta_{mm'}^{\mathrm{Kr}}$.

can be given, namely

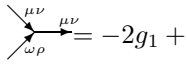


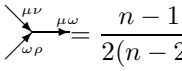
;

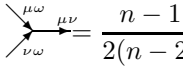
inward (outward) arrows correspond to ordinary (reciprocal) basis functions, respectively. Such vertices have the remarkable property, a kind of a *selection rule*, that the replica numbers attached to the mode m''[11] must occur either in m or in m'; otherwise the vertex is zero.

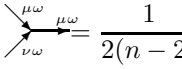
Hereinafter we give a list of the nonzero vertices. To confine the extent of the paper, vertices with replicon modes of type-II will also be omitted, although they are available; these vertices are necessary only for a calculation higher order than one-loop. The presentation follows the order introduced in equation (48), different symbols are used to indicate different replicas.

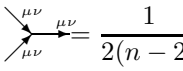
RRR:


$= -2g_1 + g_2$

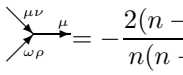

$= \dfrac{n-1}{2(n-2)}(-ng_1 + g_2)$

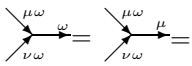

$= \dfrac{n-1}{2(n-2)}\big[(n-1)(n-4)g_1 + g_2\big]$

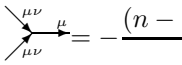

$= \dfrac{1}{2(n-2)}\big[(n^2-9n+12)g_1 - (n^2-6n+7)g_2\big]$

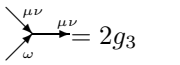

$= \dfrac{1}{2(n-2)}\big[2(3n^2-15n+16)g_1 + (n^3-9n^2+26n-22)g_2\big]$

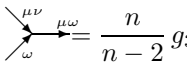
RRA:

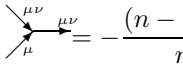

$= -\dfrac{2(n-1)^2}{n(n-2)}\,g_3$


$= \dfrac{(n-1)^2(n-4)}{n(n-2)}\,g_3$

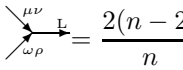

$= -\dfrac{(n-1)^2(n-3)(n-4)}{n(n-2)}\,g_3$

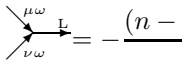

$= 2g_3$

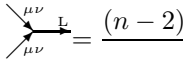

$= \dfrac{n}{n-2}\,g_3$

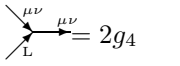

$= -\dfrac{(n-1)(n-4)}{n-2}\,g_3$

RRL:

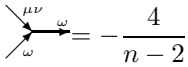

$= \dfrac{2(n-2)}{n}\,g_4$

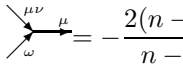

$= -\dfrac{(n-2)(n-3)}{n}\,g_4$

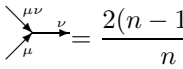

$= \dfrac{(n-2)^2(n-3)}{n}\,g_4$

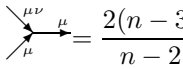

$= 2g_4$

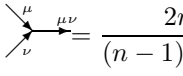
RAA:


$= -\dfrac{4}{n-2}\,g_5$

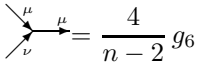

$= -\dfrac{2(n-1)}{n-2}\,g_5$

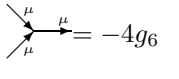

$= \dfrac{2(n-1)(n-3)}{n-2}\,g_5$


$= \dfrac{2(n-3)}{n-2}\,g_5$


$= \dfrac{2n^2}{(n-1)(n-2)}\,g_5$

AAA:

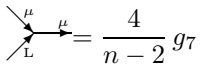

$= \dfrac{4}{n-2}\,g_6$


$= -4g_6$

[11] It is a single number (μ) if m'' is an anomalous mode, whereas replicon modes are labeled by a pair of replicas, as explained in Appendix A, ($\mu\nu$) or ($\mu\bar{\mu}$). There is, of course, no restriction if m'' is the longitudinal mode.

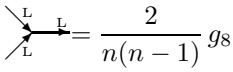
AAL:

$$= -\frac{2}{n-1}\, g_7 \qquad = 2g_7$$



$$= \frac{4}{n-2}\, g_7$$

LLL:



$$= \frac{2}{n(n-1)}\, g_8$$

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